Certified Meshing of RBF-based Isosurfaces

university of groningen

Amit Chattopadhyay amit@cs.rug.nl

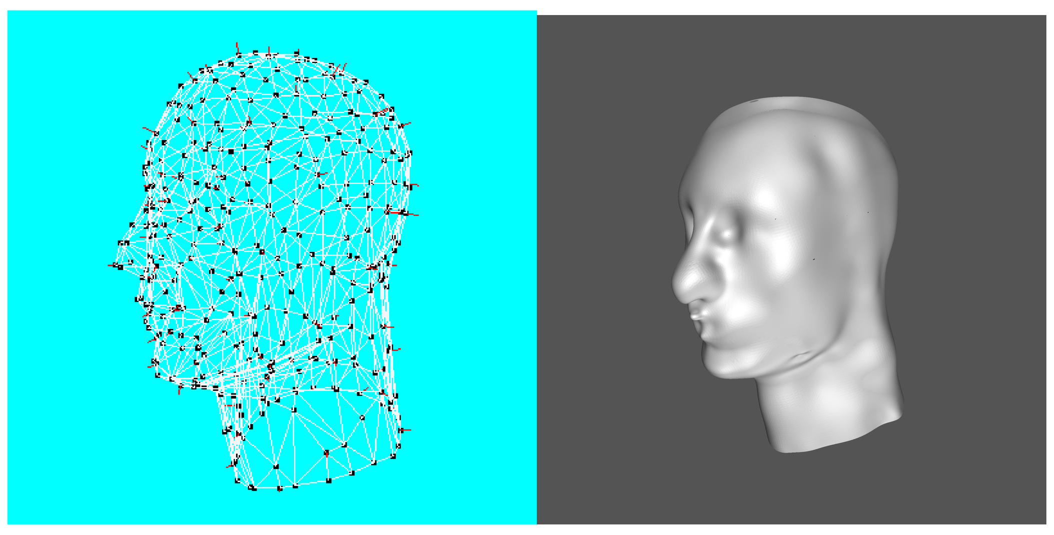
Simon Plantinga simon@cs.rug.nl

Gert Vegter G.Vegter@rug.nl

RBF Method-Shape Reconstruction

Shape Reconstruction.

- **Application:** CAD, Medical Imaging, Computer Graphics



- **Input:** Finite set of centers $V = \{v_1, v_2, \ldots, v_n\}$ sampled from a smooth surface
- **Output:** PL-approximation of the zero level set of the radial basis function (RBF) interpolant s, such that $s(v_k) = 0$, where

$$s(\underline{x}) = \sum_{k=1}^{n} w_k\,\varphi(||\underline{x} - v_k||) + p(\underline{x}).$$

- **RBF Examples:** (i) Linear: r, (ii) Triharmonic: r^3, (iii) Thin Plate Spline: $r^2 \log r$, (iv) Multiquadric: $\sqrt{1+r^2}$, (v) Gaussian: e^{-r^2} etc.

Two-step Method:

1. Constructing the RBF-interpolant s such that shape (to be reconstructed) is implicitly defined by the zero set,
2. Meshing of the zero level set of s for geometry processing.

Certified Meshing Algorithm

Certified Meshing. Approximating mesh is isotopic to the implicit surface, based on interval arithmetic.

Certified Meshing Algorithm (CMA) [6]:

ApproximateSurface (s, B)

1. Initialize octree T to box B;
2. Subdivide T until for all leaves I:

$$0 \notin \square s(I) \vee \langle \square\nabla s(I), \square\nabla s(I)\rangle > 0$$

3. BalanceOctree(T)
4. Mesh(T).

Range Function. Range Function $\square F$ for $F : \mathbb{R}^m \to \mathbb{R}^n$ computes an n-dimensional interval $\square F(I)$ for each m-dimensional interval I s.t. $F(I) \subset \square F(I)$

Predicates:

- $0 \notin \square F(I)$: Range does not contain zero,
- $\langle \square\nabla F(I), \square\nabla F(I)\rangle > 0$: Small Normal Variation.

Research Problem

- **Issue:** Performance of CMA for RBF interpolants is poor when Interval Arithmetic (IA) or Affine Arithmetic (AA) is used for computing the predicates.
- **Contribution:** Reducing Space and Time complexity of CMA by efficient computation of $\square s(I)$ and $\square s_x(I), \square s_y(I), \square s_z(I)$ for RBF Interpolant s

Example :

To find the range interval of the sum of $f(x) = x^2$ and $g(x) = (1-x)^2$ over $I = [0, 1]$.

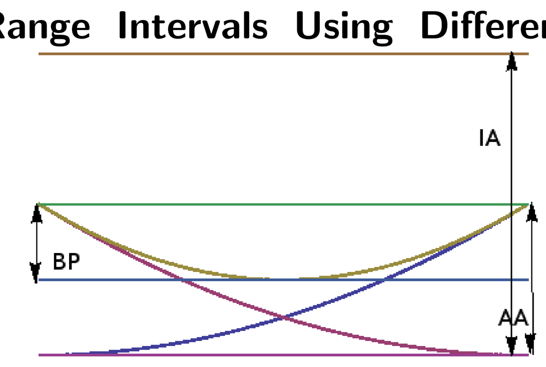
- The actual range interval $[0.5, 1]$.
- **Using IA.** $\square f(I) + \square g(I) = [0, 1] + [0, 1] = [0, 2]$
- **Using AA.** First, the variable x is expressed in affine form (AF) $\hat{x} = 0.5 + 0.5\epsilon_1$. Then, affine forms corresponding to $f(x)$ and $g(x)$ are computed as $\hat{z}_1 = 0.25 + 0.5\epsilon_1 + 0.25\epsilon_2$ and $\hat{z}_2 = 0.25 - 0.5\epsilon_1 + 0.25\epsilon_2$. Therefore, the AF corresponding to sum is $\hat{z} = 0.5 + 0.0\epsilon_1 + 0.25\epsilon_2 + 0.25\epsilon_3$. Here, ϵ_i s are symbolic variables whose values are in interval $[-1, +1]$. Hence, the range interval corresponding to the sum is $[0, 1]$.

Strategies

Bounding Plane Bounding Quadric (BPBQ).

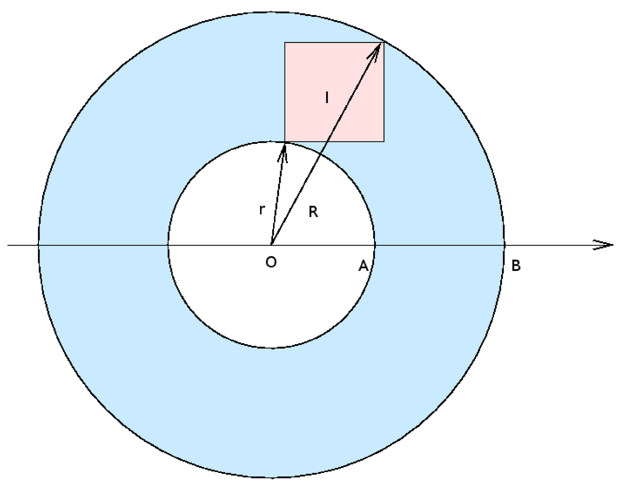
- Find upper and lower bounding planes corresponding to graph of each component function.
- Add respective linear functions to obtain a linear lower and upper bound for computing $\square F(I)$.
- A quadratic upper and lower bound is computed for computing $\square\nabla F(I)$.

Example: Range Intervals Using Different Strategies.



Bounding Paraboloid (BParab) for RBF.

- Based on the observation that the summand of the RBF-interpolant $w_k\varphi(\|\underline{x} - v_k\|)$ is radially symmetric with respect to the center v_k.
- Find upper and lower bounds of the form $\alpha_k r^2 + \beta_k$ for the univariate function $w_k\varphi(r)$ for r ranging over the smallest interval $J_k = [r_1, r_2]$ for which $r_1^2 \leq \|\underline{x} - v_k\|^2 \leq r_2^2$, for all $\underline{x} \in I$.
- Obtain quadratic upper and lower bound of s over I by summing them up.
- Compute the maximum and minimum values of these upper and lower bounds over I, say $U(I)$ and $L(I)$, respectively. Thus $\square s(I) = [L(I), U(I)]$.
- Computing $\square\nabla s(I)$ is similar.



Subdivision. To improve the performance even further a hybrid approach has been designed in which the BParab and BPBQ strategies are extended by a preliminary subdivision of the boxes. Since $\cup_{i=1}^{n}\square s(I_i) \subseteq \square s(\cup_{i=1}^{n} I_i)$, the range intervals $\square s(I)$ and $\square\nabla s(I)$ might become smaller by first subdividing the interval I into n subintervals $I_1, \ldots, I_n$, followed by computing the range intervals $\square s(I_i)$ for each subinterval I_i.

Convergence. The approximation error in the range interval using the AA, BPBQ or BParab strategy depend quadratically on the size of the input interval, where as, for IA this dependency is linear.

Results

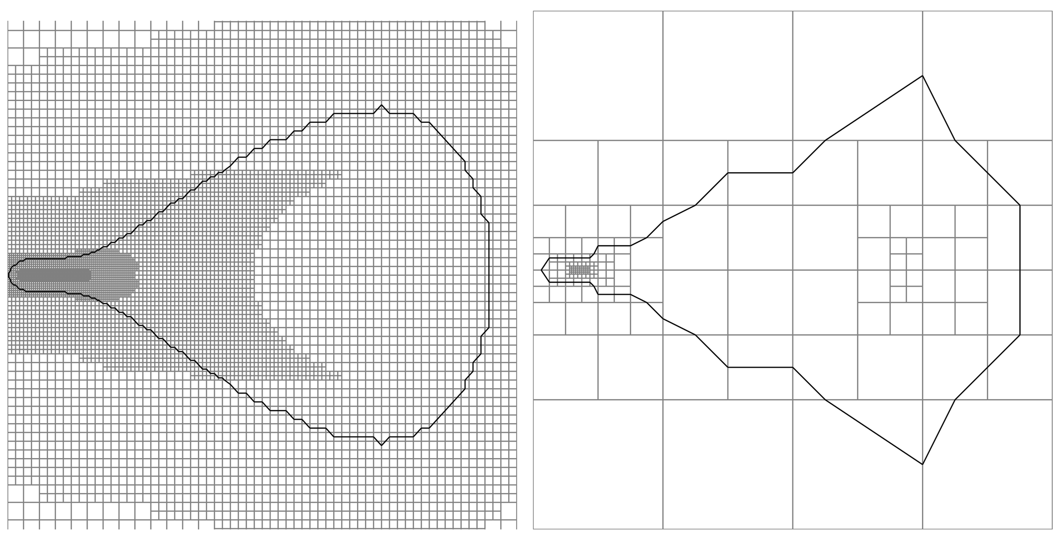


Figure 4: *Isocurve extraction for a multiquadric RBF-interpolant (49 centers) of the function $4y^2 - (x+1)^3(1-x)$, sampled uniformly on the square $[-1.1, 1.1] \times [-1.1, 1.1]$ using strategies: (i) AA and (ii) BPARAB.*

Comparison of Space and Time Complexity.

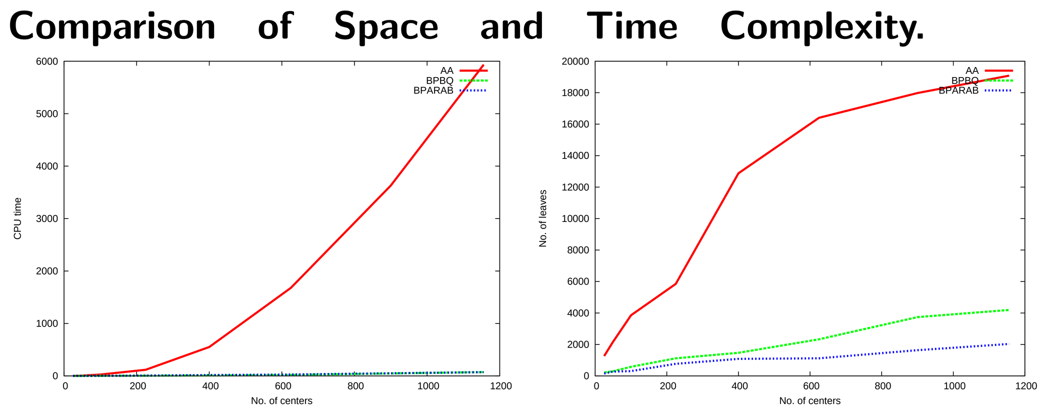


Figure 5: *(i)number of centers vs. CPU-time, (ii)number of centers vs. number of leaves of the subdivision tree*

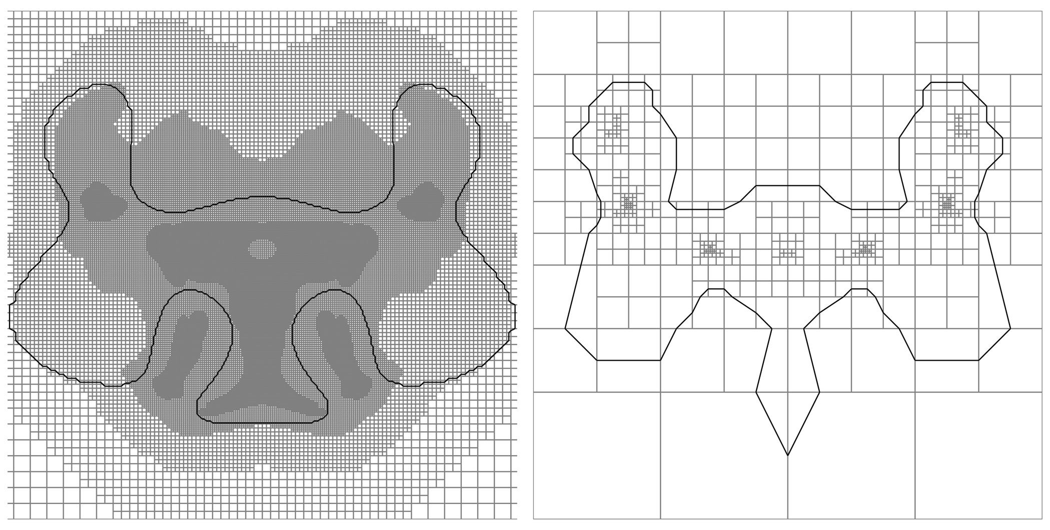


Figure 6: *Isocurve extraction for a multiquadric RBF-interpolant (25 centers) of the function $(y - x^2 + 1)^4 + (x^2 + y^2)^4 - 1 = 0$, sampled uniformly on the square $[-1.2, 1.2] \times [-1.4, 1.0]$ using (i) AA and (ii) BPARAB.*

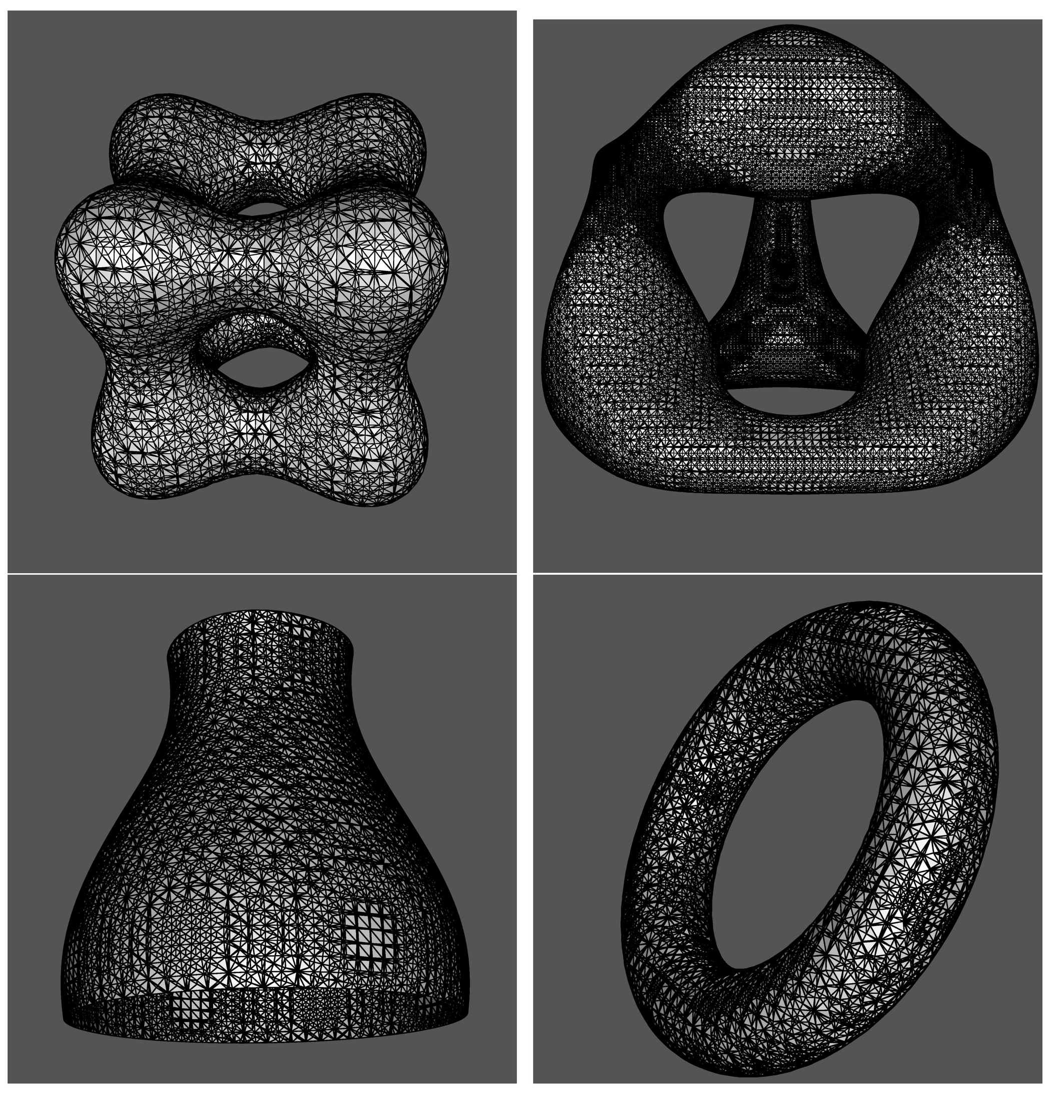


Figure 7: *Meshing of some RBF-based Isosurfaces (using BParab Strategy).*

Conclusion

IA has unacceptable performance, AA converges in most experiments but its performance is poor. The BParab-strategy works efficiently for almost all well-known RBFs, and the BPBQ-strategy gives comparable results for cubic RBFs.

Open Questions.

- To find a good sampling criteria such that the zero level set of the corresponding RBF-interpolant is certified.
- Instead of subdividing the whole domain of the implicit function, one could try to find a subset containing the zero set of the implicit function.
- Finally, the range $\square s_x(I)\square s_x(I) + \square s_y(I)\square s_y(I)$ is a superset of $\langle \square\nabla s(I), \square\nabla s(I)\rangle$. Therefore, there is enough opportunity for improving the performance of the meshing algorithm.

References

[1] J.-D. Boissonnat, D. Cohen-Steiner, and G. Vegter. Isotopic implicit surface meshing. *Discrete and Computational Geometry*, 39:138–157, 2008.

[2] A. Chattopadhyay, S. Plantinga, and G. Vegter. Certified Meshing of RBF-based Isosurfaces. *Abstracts of the 25th European Workshop on Computational Geometry, Brussels, Belgium*, 101-104, 2009.

[3] L. H. de Figueiredo and J. Stolfi. Affine arithmetic: Concepts and applications. *Numerical Algorithms.*, 00:1–13., 2003.

[4] A. Iske. Scattered data modelling using radial basis functions. In A. Iske, E. Quak, and M. S. Floater, editors, *Tutorials on Multiresolution in Geometric Modelling*, Mathematics and Visualization, pages 287–315. Springer-Verlag, Heidelberg, 2002.

[5] R. Martin, H. Shou, I. Voiculescu, A. Bowyer, and G. Wang. Comparison of interval methods for plotting algebraic curves. *Comput. Aided Geom. Des.*, 19(7):553–587, 2002.

[6] S. Plantinga and G. Vegter. Isotopic meshing of implicit surfaces. *The Visual Computer*, 23:45–58., 2007.