Certified Computation of planar Morse-Smale Complexes

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Abstract

The Morse-Smale complex is an important tool for global topological analysis in various problems of computational geometry and topology. Algorithms for Morse-Smale complexes have been presented in case of piecewise linear manifolds [3]. However, previous research in this field is incomplete in the case of smooth functions. In the current paper we use interval arithmetic to compute topologically correct Morse-Smale complex of smooth functions of two variables. The algorithm can also compute geometrically accurate Morse-Smale complex.

1 Introduction

Problem statement. A Morse function $h : \mathbb{R}^2 \to \mathbb{R}$ is a real-valued function with non-degenerate critical points (i.e., critical points with non-singular Hessian matrix). As is well-known, non-degenerate critical points are either maxima, or minima, or saddle points. We are interested in the configuration of integral curves of the *gradient vector field* ∇h of h. A *stable (unstable) separatrix* of a saddle point is the set of all regular points whose forward (backward) integral curve flows into the saddle point. (This notion will be made more precise in Section 2.) A *Morse-Smale function* is a Morse function whose stable and unstable separatrices are disjoint. In particular, the unstable separatrices flow into a sink (maximum), and the stable separatrices flow into a source (minimum). The corresponding gradient vector field will be called a *Morse-Smale system* (MS-system). A *Morse-Smale complex* (MS-complex for short) consists of all separatrices corresponding to a *MS-system.* The MS-complex describes the global structure of a Morse-Smale function. We consider the problem of computing a *certified* approximation of the MS-complex of a Morse-Smale function, i.e., a configuration of curves that is isotopic to the MS-complex. Our algorithm is based on interval arithmetic.

Our Contribution. We present an algorithm computing such a certified approximation of the MS-complex of a given smooth Morse-Smale function on the plane. In particular, the algorithm determines

- isolated certified boxes for saddles, sources and sinks.
- certified initial and terminal intervals for saddle-source or saddle-sink connectors (separatrices).
- disjoint strips around each separatrix, which can be as close to the separatrix as desired.

Related Work. Computing Morse-Smale complexes has been widely studied for piecewise-linear functions [3]. Computing MS-complexes is strongly related to vector field visualization [4]. In a similar context, designing of vector field on surfaces has been studied for many graphics applications [6]. The survey paper [2], focussing on geometrical-topological properties of real functions, gives an overview of recent work on MS-complexes.

2 Preliminaries

Morse function. A function $h : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}$ is called a *Morse function* if all its critical points are non-degenerate. The *Morse lemma* [3] states that near a non-degenerate critical point a it is possible to choose local co-ordinates x, y in which h is expressed as $h(x, y) = h(a) \pm x^2 \pm y^2$. The number of minus signs is called the index $i_h(a)$ of h at a. Thus a two variable Morse function has three types of non-degenerate critical points: minima (index 0), saddles (index 1) and maxima (index 2).

Integral line. An integral line $x : I \subset \mathbb{R} \to \mathcal{D}$ passing through a point p_0 on $\mathcal{D}$ is the unique maximal curve satisfying: $\dot{x}(t) = \nabla h(x(t)),\ x(0) = p_0$, for all $t \in I$. Integral lines corresponding to the gradient vector field of a smooth function $h : \mathcal{D} \to \mathbb{R}$ have many interesting properties, such as: (1) any two integral lines are either disjoint or coincide; (2) an integral line $x : I \to \mathcal{D}$ through a point p of h is injective and if $\lim_{t \to \pm\infty} x(t)$ exists, it is a critical point of h; (3) the function h is strictly increasing along the integral line of a regular point of h and integral; (4) regular integral lines are perpendicular to regular level sets of h.

Stable and unstable manifolds. Consider the integral line $x(t)$ passing through a point p. If the limit $\lim_{t\to\infty} x(t)$ exists, it is called the ω-limit of p and is denoted by $\omega(p)$. Similarly, $\lim_{t\to-\infty} x(t)$ is called the α-limit of p and is denoted by $\alpha(p)$ – again provided this limit exists. The stable manifold of a singular point p is the set $W^s(p) = \{q \in \mathcal{D} \mid \omega(q) = p\}$. Similarly, the unstable manifold of a singular point p is

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the set $W^u(p) = \{q \in \mathcal{D} \mid \alpha(q) = p\}$ The stable (unstable) manifolds of a saddle point (not including the saddle point itself) are called the stable (unstable) separatrices of the saddle point. Each saddle has two stable and two unstable separatrices.

The Morse-Smale complex. A Morse function on $\mathcal{D}$ is called a Morse-Smale (MS) function if its stable and unstable separatrices are disjoint. The MS-complex associated with a MS-function h on $\mathcal{D}$ is the subdivision of $\mathcal{D}$ formed by the connected components of the intersections $W^s(p) \cap W^u(q)$, where p, q range over all singular points of h. According to quadrangle lemma [3], each region of the MS-complex is a quadrangle with vertices of index $0, 1, 2, 1$, in this order around the region.

Poincaré-Hopf Index Theory. Suppose we have a vector field over some simply connected domain $\mathcal{D}$ in the two-dimensional plane. Let Γ be any closed loop in $\mathcal{D}$ which does not pass through any fixed point of the vector field. Now, as we move around Γ in the counter-clockwise sense (which is taken as the positive direction), the vectors on Γ rotate, and when we get back to the starting-point, they will have rotated through an angle $2\pi i_\Gamma$, where i_Γ is an integer, called the *Poincaré-Hopf index* [5] (or, *index* for short) of Γ. For the gradient vector field $\nabla h \equiv (h_x, h_y)$ the index i_Γ of a closed curve Γ, is found by:

$$i_\Gamma = \frac{1}{2\pi} \oint_\Gamma d\phi = \frac{1}{2\pi} \oint_\Gamma d(\tan^{-1} \frac{h_y}{h_x}). \tag{1}$$

The index of a critical point, say p, of a vector field, say X, is denoted by $i_X(p)$ and is defined to be the index i_Γ of a closed curve Γ which contains only the critical point p, and where no other critical points are on the closed curve. The following result is well-known in Index Theory.

Theorem 1 *(i) The index of a sink and a source is* $+1$.

(ii) The index of a saddle point is -1.

(iii) The index of a closed curve not containing any critical point is 0.

(iv) The index of a closed curve is equal to the sum of the indices of the fixed points within it.

Let p be a critical point of a Morse-function h, say with index $i_h(p)$. Then p is also critical point of the gradient vector field of h, say with Poincaré-Hopf index $i_{\nabla h}(p)$. Then $i_{\nabla h}(p) = (-1)^{i_h(p)}$.

Interval Arithmetic (IA). Interval arithmetic is used to prevent rounding errors in finite precision computations. A *range function* $\Box F$ for a function $F : \mathbb{R}^m \to \mathbb{R}^n$ computes for each m-dimensional interval I (i.e., an m-box) an n-dimensional interval $\Box F(I)$, such that $F(I) \subset \Box F(I)$. A range function is said to be *convergent* if the diameter of the output interval converges to 0 when the diameter of the input interval shrinks to 0. Convergent range functions exist for the basic operators and functions, so all range functions are assumed to be convergent.

3 Methods and Results

Computing the MS-complex of a Morse-Smale function $h : \mathcal{D} \mapsto \mathbb{R}$ reduces to computing separatrices of the corresponding gradient system. More precisely, for computing a certified MS-complex of a Morse function h over $\mathcal{D}$ we proceed as follows:

1. Compute certified intervals of the critical points and to detect their types corresponding to the MS-function.
2. Compute guaranteed one-dimensional intervals corresponding to initial points of each of the separatrices.
3. Compute certified bounds of the separatrices starting from one of these one-dimensional intervals to the correct source or sink.

3.1 Local Analysis: Isolating Critical Points.

The following subdivision algorithm isolates the critical points of a MS-function function h over a bounding box B ($\subseteq \mathcal{D}$). Moreover, the type of each critical point in the corresponding interval is also determined by index and orientation test. We consider the following assumption.

Assumption A: Given a function h we can find a positive number ϵ_c such that in any interval I (from the domain of h) of diameter less than ϵ_c, h can have at most one critical point inside I.

Algorithm. SEARCHCRITICAL(h, B)

1. Initialize a quadtree $\mathcal{T}$ to the bounding square B.
2. Subdivide $\mathcal{T}$ until for all the leaves I we have:

$$\underbrace{0 \notin \Box h_x(I)}_{(i)} \vee \underbrace{0 \notin \Box h_y(I)}_{(ii)} \vee \underbrace{diam(I) < \epsilon_c}_{(iii)}.$$

3. For each leaf I
4. Do if $\neg(i)$, $\neg(ii)$ and (iii) hold then
5. Compute $i_\Gamma :=$ index of boundary Γ of I
6. If $i_\Gamma = 0$
7. h has no critical point inside I
8. If $i_\Gamma = 1$
9. h has a source/sink inside I
10. If $i_\Gamma = -1$
11. h has a saddle inside I

Computing the index of a contour. We assume the curve Γ contains at most one critical point strictly inside it. Now, using the formula in (1) the computation of the index over the rectangular contour Γ boils down to finding all the "jump"-discontinuities of the function $\tan^{-1} \frac{h_y}{h_x}$ over Γ and adding them up. In other words, this reduces to finding isolated 1D-intervals, say J_{c_i} (Figure1), corresponding to zeros c_i of h_x parameterized over Γ such that on these intervals the sign of h_y does not change. Now depending on the change of sign of $\frac{h_y}{h_x}$ over J_{c_i}, while traversing Γ anticlockwise sense, the contribution in the integration is $-\pi$ (when the change of sign is from negative to positive) or $+\pi$ (when the change of sign is from positive to negative).

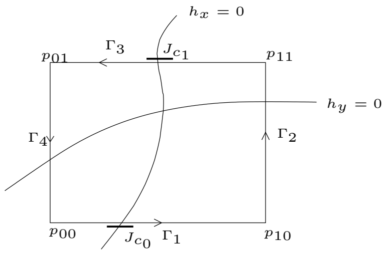

Figure 1: Index of the rectangle $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

Certified Intervals for Sinks and Sources. Note that in the algorithm SearchCritical(h, B) we use index test to distinguish a saddle from a source or a sink. However, index test cannot distinguish a source from a sink. One method to do so is by orientation test (see in Figure 3). The orientation of the gradient vector field on the boundary of the interval containing the isolated sink is inwards, whereas for a source it is outwards. Again we note that an isolated interval for a source or a sink, obtained using algorithm SearchCritical may not be certified. Here by certified interval we mean any integral curve entering into the interval eventually meet the sink or source in forward or backward time (without leaving the box). However, the previous algorithm does not exclude the possibility of having intervals as in Figure 2. One

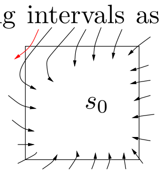

Figure 2: An isolated interval for sink s_0. The interval is not certified.

approach to find a certified interval is by subdividing the interval I recursively, until we find a subinterval such that the orientation of the gradient field on its boundary is either completely inward or completely outward (Figure 3).

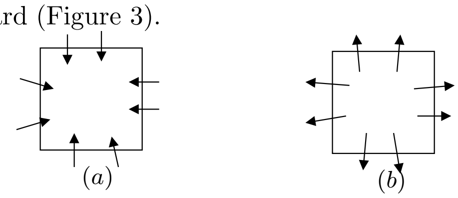

Figure 3: Orientation of the vector field on the boundary of an interval for a (a) sink, (b) source.

Local analysis of saddle intervals. Algorithm SearchCritical(h, B) computes isolated 2D-intervals corresponding to each saddle point of h. Now we find four disjoint one-dimensional intervals on the boundary of the 2D-interval (figure 4) such that each of the four separatrices passes through one of these 1D-intervals. The method consists of following three steps.

1. First, we refine the box containing the saddle, recursively, until the function restricted over the boundary of the box has exactly four extrema (two maxima and two minima), Figure 4-(a).
2. Next we find four disjoint 1D-intervals containing four extrema, respectively Figure 4-(b).
3. The final step is to refine these four 1D-intervals, recursively, until they are guaranteed to contain the corresponding separatrix. This requires to satisfy an orientation property between two consecutive 1D-intervals, Figure 4-(c).

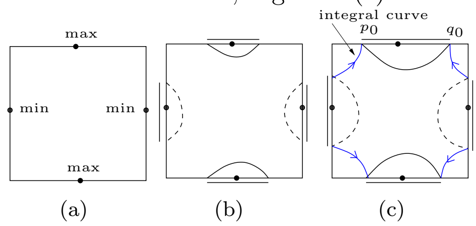

Figure 4: Saddle box: (a) refining until four extrema,(b) four initial 1D-intervals, (c) refining 1D-intervals by orientation test.

3.2 Global Algorithm: Certified Separatrices

Finally, corresponding to each separatrix we compute a certified region bounded by two piecewise linear boundaries. The piecewise linear boundaries start from the end points of the 1D-intervals near each saddle. We need the following assumption for the convergence of the algorithm.

Assumption B: ψ-normal variation. We assume that the function h satisfies ψ-*normal variation* condition, which is: for $x_1, x_2 \in \mathcal{D} \setminus \mathcal{C}$, $\|x_1 - x_2\| \le \delta$ and $\delta > 0$,

$$\frac{\langle \nabla h(x_1), \nabla h(x_2) \rangle}{\|\nabla h(x_1)\| \|\nabla h(x_2)\|} > \cos \psi.$$

Here, $\mathcal{C}$ denotes the union of all certified intervals of the critical points of h in $\mathcal{D}$.

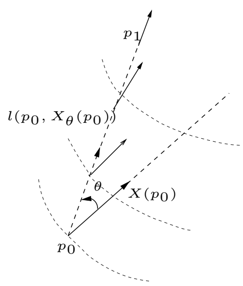

Figure 5: Line-segment $\overline{p_0 p_1}$ on which orientation and monotonicity property hold.

Lemma 2 *Let $X := \nabla h$ satisfies ψ-normal variation in $\mathcal{D} \setminus \mathcal{C}$. Again let, X_θ denote the corresponding vector field rotated anti-clockwise by an angle θ ($\psi \le \theta < \frac{\pi}{2}$). Then for a directed line segment $l(p_0, X_\theta(p_0))$ along the direction $X_\theta(p_0)$ with starting point p_0 and length δ (segment $\overline{p_0 p_1}$, in the figure 5), the following properties hold:*

(i) for each point $q \in l(p_0, X_\theta(p_0))$, $\Delta(X(q), X_\theta(p_0)) > 0$, (here $\Delta(v_1, v_2) := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ for $v_i \equiv (a_i, b_i)$)

(ii) h is monotonically increasing along this line segment $l(p_0, X_\theta(p_0))$.

(iii) The minimum change of the height function h on this line segment is $\delta \cos\psi$.

Algorithm: ComputeSeparatrixBounds(h, B_0)
Input: Vector-field X, starting 1D-interval of the separatrix $\overline{p_0q_0}$, bounding box B_0.
Output: A certified stripe containing a separatrix.

1. Initialize: $\theta \leftarrow \theta_0 (\theta_0 < \frac{\pi}{2})$
2. Rotate(X, θ): Rotate vector-field $X(p)$ anticlockwise by an angle θ.
3. ComputeLeftBoundary(p_0, X_θ): Compute polygonal line-segment l_θ starting from the point p_0 such that the separatrix (passing through the line segment $\overline{p_0q_0}$) remains on the right side of l_θ.
4. Rotate$(X, -\theta)$: Rotate vector-field $X(p)$ clockwise by an angle θ.
5. ComputeRightBoundary$(q_0, X_{-\theta})$: Compute polygonal line-segment $l_{-\theta}$ starting from the point q_0 such that the separatrix (passing through the line segment $\overline{p_0q_0}$) remains on the left side of $l_{-\theta}$.
6. **if** both l_θ and $l_{-\theta}$ meet the same "sink" (or "source") and the region between l_θ and $l_{-\theta}$ does not contain any critical point, then the separatrix converges to that sink; **return**.
7. **else-if** both l_θ and $l_{-\theta}$ meet the boundary of the box B_0 and the region between l_θ and $l_{-\theta}$ does not contain any critical point, then the separatrix converges to the boundary; **return**.
8. **else** $\theta \leftarrow \frac{\theta}{2}$ and **goto** step 2.

Convergence. To prove that the algorithm converges in finite number of steps, we prove the following. First, using lemma (2) we find an upper bound of the number of segment in the line l_θ.

Theorem 3 *Let the change of the height function h along a separatrix, outside the critical region, be H (computed as: $h(final) - h(starting)$). Then an upper bound of the number of segments in l_θ is given by:* $\left\lceil \frac{H}{\delta \cos\psi} \right\rceil$.

This proves that the sub-procedures: ComputeLeftBoundary and ComputeRightBoundary converge in a finite number of steps. Moreover, using the assumption B, we can prove that the global algorithm converges in a finite number of steps.

4 Implementation Results

In this section we illustrate a few implementation outputs with timing results of our algorithm. In figures 6-8 we compute the certified MS-complex of different functions for distinct values of the parameters ϵ_c (described in the algorithm SearchCritical) and angle θ of rotation of the vector field. We use the `Boost` library [1] for IA. All experiments have been performed on a 3GHz Intel Pentium 4 machine under Linux with 1 GB RAM using the `g++` compiler, version 3.3.5.

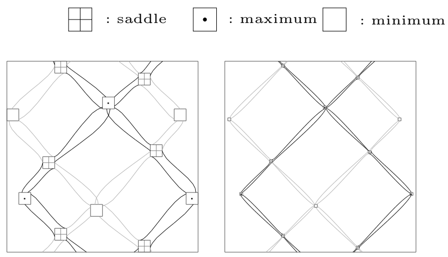

Figure 6: MS-Complex of the function: $\cos x \ \sin y + 0.2\,(x+y)$ for (i) $\epsilon_c = 0.5$, $\theta = \frac{\pi}{10}$, CPU-time = 8 sec., (ii) $\epsilon_c = 0.2$, $\theta = \frac{\pi}{30}$, CPU-time = 20 sec., inside box $[-3.5, 3.5] \times [-3.5, 3.5]$.

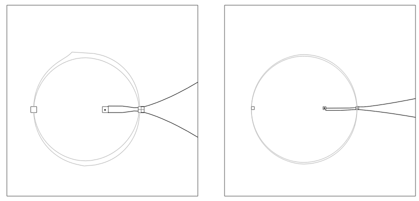

Figure 7: MS-Complex of the function: $10x - \frac{13}{2}(x^2+y^2) + \frac{1}{3}(x^2+y^2)^2$ for (i) $\epsilon_c = 0.5$, $\theta = \frac{\pi}{10}$, CPU-time = 0.16 sec., (ii) $\epsilon_c = 0.2$, $\theta = \frac{\pi}{30}$, CPU-time = 0.5 sec., inside box $[-5, 6] \times [-5, 6]$.

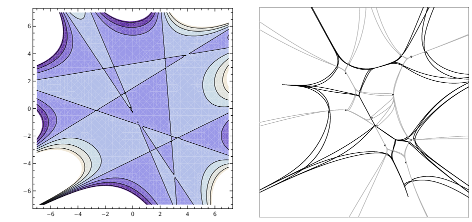

Figure 8: MS-Complex of the function constructed by multiplying 7 linear functions: (i) contour plot using Mathematica, (ii) MS-complex with $\epsilon_c = 0.17$, $\theta = \frac{\pi}{30}$, CPU-time = 15 min., inside box $[-7, 7] \times [-7, 7]$.

Conclusion. The outcome of our research is twofold. Firstly, we compute the topologically correct MS-complex of a Morse-Smale system. The saddle-sink or saddle-source connectivity can also be represented as a graph. On the other hand, depending on a user-specified parameter we can compute the geometrically accurate MS-complex.

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